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Approximation of Bounded Sets*

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In this paper, we give a unified treatment of the problem of approximating a family of elements belonging to a space of real-valued functions simultaneously by a single element from a specified approximating family. Specifically, if F denotes a uniformly bounded subset of a linear vector space X with norm $\| \ \|$, and V denotes a nonempty convex subset of X, we seek an element $v_0 \in V$, designated a best simultaneous approximation (b.s.a.), assuming it exists, satisfying

$$\sup_{f \in F} \|f - v_0\| = \inf_{v \in V} \sup_{f \in F} \|f - v\|. \tag{1}$$

For example, if $X = C[a, b]$, the space of continuous real functions defined on $[a, b]$ and endowed with the uniform norm ($\|g\| = \sup_{x \in [a,b]} |g(x)|$ for all $g \in C[a, b]$), $F = \{f_1, f_2\} \subset X$ and $V = P$, the polynomials of degree not greater than some fixed integer, then we seek $p_0 \in P$ such that

$$\max\{\|f_1 - p_0\|, \|f_2 - p_0\|\} = \inf_{p \in P} \max\{\|f_1 - p\|, \|f_2 - p\|\}.$$

This problem has been studied in [6].

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The case of $X = S[a, b]$, the space of bounded real functions defined on $[a, b]$ and endowed with the uniform norm, F a bounded, not necessarily finite family in $S[a, b]$ and V a linear subspace of $C[a, b]$, has been studied in [3].

The problem of approximating in the sense of (1) for a general normed linear space X by elements of a closed convex subset, such as when convex side constraints are imposed on a linear approximating family, was first tackled in [8], by the method of subgradients. In particular, the results obtained there relate to when F is a compact subset of X, while it would be desirable to extend the setting to the less restrictive condition of F being uniformly bounded in norm. This we do in Sections 2 to 4 by a direct approach to the problem based on the Hahn–Banach theorem and generalizations of concepts in [3].

In Sections 5 and 6, respectively, we derive from our unifying theory, the linear unconstrained problem of simultaneous approximation of F when

(i) F consists of a bounded set in $S[a, b]$, and

(ii) F consists of an upper semicontinuous real-valued function f^+ and a lower semicontinuous function f^-, with $f^+ \geqslant f^-$ pointwise over a compactum.

Conditions under which the existence of the b.s.a. is guaranteed, have been given in [4, 7, 11].

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Let X be a normed vector space and X^* the real dual space of bounded linear functionals on X. We let B^* denote the unit norm ball on X^* with $B^* := \{L \in X^*: \|L\| \leqslant 1\}$. There exists a unique smallest topology of open sets for X^* generated by any nonempty set $M \subset X$ such that all evaluation mappings $\hat{x}: X^* \to R$ given by

$$\hat{x}(L) \equiv Lx \qquad \text{for all } x \in M, \text{ are continuous on } X^*.$$

In the sequel we shall assume that X^* is endowed with such a $\sigma(M, X^*)$ topology for some appropriate $M \subset X$, where M includes at least the set V. The description of continuous or upper semicontinuous (u.s.c.) to any subset of X^* is understood to relate to the open sets of $\sigma(M, X^*)$. For example, $g(L)$ defined on $K \subset X^*$, is u.s.c. if for each real number r, $\{L \in K: g(L) < r\}$ is an open set that belongs to $\sigma(M, X^*)$. We note that when $M = X$, the topology on X^* is the weak * topology. Furthermore, since B^* is weak * compact and $M \subset X$, we may deduce that B^* is compact in the $\sigma(M, X^*)$ topology.

Now assume that K is a subset of B^* satisfying

(1) K is $\sigma(M, X^*)$ compact.

(2) For every $f \in F$ and $v \in V$, there exists a sequence $\{L\} \in K$ such that $\sup_{L \in K} L(f-v) = \|f-v\|$.

We note, in particular, that by the Hahn–Banach Theorem there always exists an $L \in B^*$ such that $L(f-v) = \|f-v\|$. In conditions (1) and (2) above, however, we do not impose on K that it must contain any such L.

We define the following functions on K. For each $v \in V$, $v(L) := Lv$, and $U_F(L) := \sup_{f \in F} Lf$. Then for each $v \in V$ we have

$$\begin{aligned}\sup_{f \in F} \|f-v\| &= \sup_{f \in F} \sup_{L \in K} [L(f-v)] \\ &= \sup_{L \in K} \sup_{f \in F} [Lf - Lv] \\ &= \sup_{L \in K} [U_F(L) - v(L)].\end{aligned}$$

Thus for $v_0 \in V$, $\sup_{f \in F} \|f-v_0\| = \inf_{v \in V} \sup_{f \in F} \|f-v\|$ if and only if

$$\sup_{L \in K} [U_F(L) - v_0(L)] = \inf_{v \in V} \sup_{L \in K} [U_F(L) - v(L)]. \qquad (2)$$

The problem of finding $v_0 \in V$ which best approximates F in the sense of (1) is thus the same as that of finding $v_0 \in V$ which best approximates the function $U_F(L)$ in the sense of (2). We may think of V as both a subset of X and as a subset of the continuous real functions on K.

It turns out to be more convenient to characterize v_0 in terms of the "upper envelope" of $U_F(L)$ instead of $U_F(L)$ itself.

For each $L \in K$, let $N(L)$ denote the collection $\{O\}$ of all open neighborhoods in K of L. We may assume without loss of generality that $N(L)$ is a local base in K of L. Then for a bounded real-valued function g defined on K, we define

$$g^+(L) := \inf_{O \in N(L)} \sup_{k \in O} g(k), \qquad L \in K.$$

Remark 1. The function $g^+(\cdot)$ is u.s.c. on K.

Proof. Let r be a given real number and L be an arbitrary member of K satisfying $g^+(L) < r$. By the definition of $g^+(L)$, there exists an $O \in N(L)$ such that

$$\sup_{k \in O} g(k) < g^+(L) + (r - g^+(L))/2.$$

Since O is a neighborhood of every $k \in O$, it follows that $g^+(k) < r$ for all $k \in O$ or, equivalently, O is contained in $\{L \in K: g^+(L) < r\}$. Hence the result.

We also note that since v is continuous on K, both v and $-v$ are u.s.c. on K.

Remark 2. There exists an $L_0 \in K$ such that

$$g^+(L_0) = \sup_{L \in K} g^+(L).$$

Proof. The proof follows from the fact that K is $\sigma(M, X^*)$ compact and $g^+(\cdot)$ is u.s.c. (see e.g., [10, p. 140]).

Remark 3. For each $v \in V$ and for each $L \in K$

$$[g(L) - v(L)]^+ = g^+(L) - v(L).$$

Proof. Let $h > 0$ and $L_0 \in K$. Choose $N_1, N_2 \in N(L_0)$ such that

(1) $g(L) - v(L) \leqslant [g(L_0) - v(L_0)]^+ + h$, for all $L \in N_1$, and

(2) $v(L) < v(L_0) + h$, for all $L \in N_2$.

Then $N_0 := N_1 \cap N_2 \in N(L_0)$ and for all $L \in N_0$, both conditions (1) and (2) are satisfied. Thus for all $L \in N_0$,

$$g(L) < [g(L_0) - v(L_0)]^+ + v(L_0) + 2h.$$

However, $g^+(L_0) \leqslant \sup\{g(L): L \in N_0\}$. Hence $g^+(L_0) - v(L_0) \leqslant [g(L_0) - v(L_0)]^+$.

To show that the inequality can be reversed, let $h > 0$ and $N_1, N_2 \in N(L_0)$ be chosen such that

(1′) $g(L) \leqslant g^+(L_0) + h$, for all $L \in N_1$, and

(2′) $-v(L) < h - v(L_0)$, for all $L \in N_2$.

Then $N_0 = N_1 \cap N_2 \in N(L_0)$ and for all $L \in N_0$, both conditions (1′) and (2′) are satisfied. Thus for all $L \in N_0$

$$g(L) - v(L) < g^+(L_0) - v(L_0) + 2h.$$

We may take the supremum over all $L \in N_0$ of the left-hand side, and preserve the inequality.

Hence

$$[g(L_0) - v(L_0)]^+ \leqslant g^+(L_0) - v(L_0).$$

This completes the proof.

Remark 4. $\sup_{L\in K} g^+(L) = \sup_{L\in K} g(L)$.

Proof. Clearly $\sup_{L\in K} g(L) \leqslant \sup_{L\in K} g^+(L)$.

On the other hand, let $L_0 \in K$ be chosen, as in Remark 2, such that $g^+(L_0) = \sup_{L\in K} g^+(L)$. For any $h > 0$, let $L \in K$ be chosen such that $g(L) > g^+(L_0) - h$. Then

$$\sup_{L\in K} g(L) \geqslant g^+(L_0) = \sup_{L\in K} g^+(L).$$

We conclude that for each $v \in V$, $U_F^+(L) - v(L)$ is u.s.c. and further that

$$\sup_{L\in K} [U_F(L) - v(L)] = \sup_{L\in K} [U_F(L) - v(L)]^+$$
$$= \sup_{L\in K} [U_F^+(L) - v(L)].$$

At this point, we have shown that

$$\sup_{f\in F} \|f - v_0\| = \inf_{v\in V} \sup_{f\in F} \|f - v\|$$

if and only if

$$\sup_{L\in K} [U_F^+(L) - v_0(L)] = \inf_{v\in V} \sup_{L\in K} [U_F^+(L) - v(L)]. \tag{3}$$

That is to say, the original problem of approximating the set F is the same as that of approximating the u.s.c. function $U_F^+(L)$.

3

The theorems developed in this section will characterize v_0 in terms of U_F^+.

THEOREM 1. *Let K be an arbitrary compact Hausdorff space, and g a real-valued upper semicontinuous function defined on K. Let V be a convex subset of the continuous real-valued functions on K such that*

$$\inf_{v\in V} \max_{L\in K} [g(L) - v(L)] > -\infty.$$

Let $v_0 \in V$. Then

$$E := \max_{L\in K} [g(L) - v_0(L)] = \inf_{v\in V} \max_{L\in K} [g(L) - v(L)] \tag{4}$$

if and only if for each $v \in V$ there exists an $L \in K$ such that

(i) $g(L) - v(L) = E$, *and*

(ii) $v(L) - v_0(L) \leqslant 0$.

Proof. The proof uses a standard argument and is included for completeness. We show first that the conditions (i) and (ii) are necessary. The proof is by contraposition.

Let $S = \{L \in K: g(L) - v_0(L) = E\}$. Since $g(L) - v_0(L)$ is u.s.c., S is nonempty and closed in K. Suppose that (i) and (ii) are not necessary. Then for some $v \in V$

$$\inf_{L \in S} [v(L) - v_0(L)] = s > 0.$$

Let W be the set of all $L \in K$ satisfying $v(L) - v_0(L) > s/2$. Then W is an open set containing S.

If we take any t, $0 < t < 1$, then for all $L \in W$

$$g(L) - v_0(L) - t(v - v_0)(L) < E - ts/2.$$

Hence $\sup_{L \in W}\{g(L) - [v_0(L) + t(v - v_0)(L)]\} \leqslant E - ts/2 < E$.

On the compact set $K \backslash W$, $g(L) - v_0(L)$ is bounded away from E. Since $(v - v_0)(L)$ is bounded on $K \backslash W$, there exists $t_0 > 0$ such that for all t, $0 \leqslant t \leqslant t_0$,

$$\sup_{L \in K \backslash W} \{g(L) - [v_0(L) + t(v - v_0)(L)]\} < E.$$

Thus for all $t \in (0, \min\{1, t_0\})$

$$\sup_{L \in K} \{g(L) - [v_0(L) + t(v - v_0)(L)]\} < E.$$

That is, $(1 - t) v_0 + tv \in V$ is a better approximation in the sense of (4) than v_0.

The argument that (i) and (ii) are sufficient is straightforward and is omitted. We note that we can replace g in Theorem 1 by U_F^+, and this we do in Sections 5 and 6. However, here we proceed to refine Theorem 1 by first deriving a significant property of the function U_F^+.

LEMMA 1. *U_F^+ is a convex function on K.*

Proof. Let $L_1, L_2 \in K$ and N_i be an open neighborhood in K of L_i for $i = 1, 2$. For any $k_1 \in N_1$, $k_2 \in N_2$ and $t \in [0, 1]$, $tN_1 + (1 - t) k_2$ is an open neighborhood of $tk_1 + (1 - t) k_2$ that is contained in $tN_1 + (1 - t) N_2$. Hence $tN_1 + (1 - t) N_2$ is an open neighborhood of $tL_1 + (1 - t) L_2$. Thus

$$\begin{aligned} U_F^+(tL_1+(1-t)L_2) &= \inf_{O\in N(tL_1+(1-t)L_2)} \sup_{L\in O} U_F(L) \\ &\leqslant \sup_{L\in tN_1+(1-t)N_2} U_F(L) \\ &= \sup_{k_1\in N_1, k_2\in N_2} U_F(tk_1+(1-t)k_2) \\ &\leqslant t\sup_{k_1\in N_1} U_F(k_1)+(1-t)\sup_{k_2\in N_2} U_F(k_2). \end{aligned}$$

Taking the infimum over $N_1 \in N(L_1)$ and $N_2 \in N(L_2)$ yields

$$U_F^+(tL_1+(1-t)L_2) \leqslant tU_F^+(L_1)+(1-t)U_F^+(L_2).$$

Now for any set A, let $\operatorname{ext}(A)$ denote the set of extreme points of A. That is, $a \in \operatorname{ext}(A)$ if and only if a cannot be expressed as a strict convex combination of any other points in A. We derive:

LEMMA 2. *Let* $E := \max_{L\in K}[U_F^+(L)-v_0(L)]$ *and let* $A(v_0) := \{L \in K: U_F^+(L)-v_0(L)=E\}$. *Then* $\operatorname{ext}(A(v_0)) = \operatorname{ext}(K) \cap A(v_0)$.

Proof. We show that $\operatorname{ext}(A(v_0)) \subset \operatorname{ext}(K) \cap A(v_0)$ since inclusion the other way follows by definition of $\operatorname{ext}(A(v_0))$.

Suppose $L \in A(v_0)$ and $L \notin \operatorname{ext}(K)$. Then $L = tL_1+(1-t)L_2$ for some $L_1, L_2 \in K$ and $t \in (0,1)$. By Lemma 1,

$$E = U_F^+(L)-v_0(L) \leqslant t[U_F^+(L_1)-v_0(L_1)]+(1-t)[U_F^+(L_2)-v_0(L_2)].$$

However, $U_F^+(L_i)-v_0(L_i) \leqslant E$ for $i=1,2$ and so we must have equality holding. Hence $L_i \in A(v_0)$. Consequently, $L \notin \operatorname{ext}(A(v_0))$ from which the result follows.

THEOREM 2. *Let* $v_0 \in V$ *and* $E = \sup_{f\in F}\|f-v_0\|$. *Then* v_0 *is a* b.s.a. *if and only if for each* $v \in V$ *there exists an* $L \in \operatorname{ext}(K)$ *such that*

(1) $U_F^+(L)-v_0(L)=E$, *and*

(2) $L(v-v_0) \leqslant 0$.

Proof. As in [1, Lemma 2], we have that

$$\min_{L\in A(v_0)} L(v-v_0) \leqslant 0 \qquad \text{if and only if} \qquad \min_{L\in \operatorname{ext}(A(v_0))} L(v-v_0) \leqslant 0.$$

Now apply this and Lemma 2 above and Theorem 1 yields Theorem 2.

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In [8], the subset F was assumed to be norm-compact in X. Under this assumption, Theorem 2 can be "improved." The following remarks are made for this purpose.

Remark 5. If F is a norm-compact subset of X, and $h > 0$, then there exists a finite number of elements of F, $f_1, \dots, f_n$ such that for every $f \in F$, $\min_i \|f - f_i\| < h$. Now let $f \in F$ and $f_j \in \{f_1, \dots, f_n\}$ be chosen such that $\|f - f_j\| < h$. Let $L \in K'$, an arbitrary nonempty subset of B^*. Then

$$Lf = L(f - f_j) + Lf_j \leqslant \|L\| \, \|f - f_j\| + \max_i Lf_i$$

$$< h + \max_i Lf_i.$$

Thus $\sup_{f \in F} Lf - \max_i Lf_i < h$ and hence for any $h > 0$

$$0 \leqslant U_F(L) - \max_i Lf_i < h \qquad \text{for all } L \in K$$

which is the main observation of this remark.

Remark 6. If F is norm-compact, then $U_F(L)$ is continuous on any nonempty subset K' of B^*. Consequently, $U_F^+(L) = U_F(L)$ independently of the subset K used to define $U_F^+(L)$.

Proof. Let $L_0 \in K'$ and $h > 0$. We show that there exists a neighborhood N_0 of L_0, in K', such that $|U_F(L) - U_F(L_0)| < h$ for all $L \in N_0$. Let $f_1, \dots, f_n$ be elements of F such that for all $f \in F$, $\min_i \|f - f_i\| < h/3$. Let

$$N_0 := \{L \in K' : |Lf_i - L_0 f_i| < h/3,\ 1 \leqslant i \leqslant n\}.$$

We can show that $|\max_i Lf_i - \max_i L_0 f_i\} < h/3$ for $L \in N_0$. For let $\max_i Lf_i = Lf_j$ and $\max_i L_0 f_i = L_0 f_k$. Then

$$\max_i Lf_i - \max_i L_0 f_i = Lf_j - L_0 f_k$$

$$= Lf_j - L_0 f_j + [L_0 f_j - L_0 f_k]$$

$$\leqslant Lf_j - L_0 f_j < h/3,$$

where we have used the fact that the term in brackets is non-positive. On the other hand,

$$\max_i Lf_i - \max_i L_0 f_i = [Lf_j - Lf_k] + Lf_k - L_0 f_k$$

$$\geqslant Lf_k - L_0 f_k > -h/3,$$

where we have used the fact that the term in brackets is nonnegative. Thus if $L \in N_0$,

$$|U_F(L) - U_F(L_0)| \leqslant |U_F(L) - \max_i Lf_i| + |\max_i Lf_i - \max_i L_0 f_i|$$

$$+ |\max_i L_0 f_i - U_F(L_0)|$$

$$< h/3 + h/3 + h/3 = h,$$

where we have used the main observation of Remark 5.

Remark 7. If F is norm-compact, then for every $L \in B^*$ there exists $f \in F$ such that $U_F(L) = Lf$.

The proof is a straightforward consequence of L being a norm-continuous functional.

THEOREM 3. *Let F be a norm-compact subset of X and V a convex subset of X. If $v_0 \in V$ and $E = \sup_{f \in F} \|f - v_0\|$ then v_0 is a* b.s.a. *to F if and only if for each $v \in V$ there exists an $L \in$* ext(B^*) *and an $f \in F$ such that*

(1) $L(f - v_0) = E$, *and*

(2) $L(v - v_0) \leqslant 0$.

Proof. Let $K = B^*$ and $L \in K$. Then through Remark 6, we can replace $U_F^+(L)$ in Theorem 2 with $U_F(L)$, and through Remark 7, we can replace $U_F(L)$ with Lf for some $f \in F$.

Some specific applications of Theorem 3 are considered in [8]. However, for the wider implications of a generalized "alternation" theorem, de la Vallee–Poussin theorem and strong unicity result, see [1, Theorem 4.3 et seq.].

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For Q a compact Hausdorff space with the usual topology of open sets, we take $X = S(Q)$, the space of bounded real-valued functions together with the uniform norm, and $V = P$, a linear subspace of $C(Q) \subset S(Q)$. F shall be a bounded subset of $S(Q)$. We endow the dual space $[S(Q)]^*$ with the topology of open sets generated by $M = C(Q)$. We define, using open neighborhoods in Q,

$$F^+(q) := (\sup_{f \in F} f(q))^+ \qquad \text{and} \qquad F^-(q) := (\inf_{f \in F} f(q))^-, \qquad q \in Q$$

where the superscript "+" has been defined above and the superscript "−" is defined as follows: For $h(q)$ a bounded real-valued function on Q,

$$h^-(q) := \sup_{N\in N(q)} \inf_{q'\in N} h(q'), \qquad q\in Q,$$

where $N(q)$ denotes the collection $\{N\}$ of all open neighborhoods of q in Q. We observe that $-h^-(q) = (-h(q))^+$, $q\in Q$.

Now for each $q\in Q$, we define the point evaluation functional L_q by $L_q x \equiv x(q)$ for all $x\in S(Q)$. We let $H^+(Q) := \{L_q : q\in Q\}$, $H^-(Q) := \{-L_q : q\in Q\}$ and $H^0(Q) := H^+(Q)\cup H^-(Q)$. In the sequel we assume that $K = H^0(Q)$. Note that conditions (1) and (2) of Section 2 defining K are met.

Remark 8. $\operatorname{ext}(B^*)\subset H^0(Q)\subset B^*$. The proof is as given in [5, V.8.6, p. 441] with minor modification.

Now for $\sigma = +, -$ or 0, and q an arbitrary element of Q, let $N^\sigma(L_q)$ be the collection $\{N^\sigma\}$ of all basic neighborhoods of L_q in $H^\sigma(Q)$. Recall that N^σ will be a set of the form $\{L\in H^\sigma(Q)\colon |(L-L_q)x| < \varepsilon$ for $x\in\theta$; θ some finite subset of M and $\varepsilon > 0\}$.

Remark 9.

$$\inf_{N^+\in N^+(L_q)} \sup_{L\in N^+} U_F(L) = F^+(q),$$

$$\sup_{N^-\in N^-(L_q)} \inf_{L\in N^-} U_F(L) = F^-(q).$$

Proof.

$$U_F(L_q) = \sup_{f\in F} L_q f = \sup_{f\in F} f(q), \qquad q\in Q.$$

$$U_F(-L_q) = \sup_{f\in F} -L_q f = -\inf_{f\in F} L_q f = -\inf_{f\in F} f(q), \qquad q\in Q.$$

For $\sigma = +$ or $-$, the mapping $q\to\sigma L_q$ is a one-to-one homeomorphism of Q onto $H^\sigma(Q)$, which is also $\sigma(M,X)$ compact, as shown in [5, V.8.7, p. 442]. Hence the result.

Remark 10.

$$U_F^+(L_q) = F^+(q), \qquad q\in Q.$$

$$U_F^+(-L_q) = -F^-(q), \qquad q\in Q.$$

Proof. First take a $q\in Q$, and let $B^0\in N^0(L_q)$. Then there exists a corresponding basic neighborhood $B^+\in N^+(L_q)$ with $L_q\in B^+\subset B^0$. Now suppose $B^+\in N^+(L_q)$ and let B^0 be the corresponding basic neighborhood in $N^0(L_q)$. Let $h > 0$ and define $f_0\in C(Q)$ by $f_0\equiv h$ on Q. Define $B' :=$

$\{L \in H^0(Q): |Lf_0 - L_q f_0| < h\} \in N^0(L_q)$. Since $|-L_{q'} f_0 - L_q f_0| = 2h > h$, for all $q' \in Q$, we have that $-L_{q'} \notin B'$, $q' \in Q$. Hence $B' \subset H^+(Q)$. Now set $N^0 = B' \cap B^0$. $N^0 \in N^0(L_q)$ and $L_q \in N^0 \subset B^+$. It may be that $N^0 = \{L_q\}$. Thus for each $q \in Q$ and each $N^+ \in N^+(L_q)$, $U_F^+(L_q) \leqslant \sup_{L \in N^+} U_F(L)$. Hence

$$U_F^+(L_q) \leqslant \inf_{N^+ \in N^+(L_q)} \sup_{L \in N^+} U_F(L)$$

$$\leqslant \inf_{N^0 \in N^0(L_q)} \sup_{L \in N^0} U_F(L) = U_F^+(L_q).$$

From Remark 9, $U_F^+(L_q) = F^+(q)$ and similarly for the second result. Theorem 1 now becomes:

THEOREM 4. *Let Q be a compact Hausdorff space, F a bounded subset of $S(Q)$ and P a subspace of $C(Q)$. Let $p_0 \in P$ and $E = \sup_{f \in F} \|f - p_0\|$. Then p_0 is a* b.s.a. *if and only if for each $p \in P$ there exists a $q \in Q$ such that either*

(i) $F^+(q) - p_0(q) = E$ *and* $p(q) \leqslant 0$, *or*

(ii) $p_0(q) - F^-(q) = E$ *and* $p(q) \geqslant 0$.

A version of this theorem is given in [4].

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We now show how the following very special problem, which has appeared in [6] and [9] in connection with simultaneous approximation theory, can be treated within our framework. With X, P and Q defined as in Section 5, let $f^+(q)$ and $f^-(q)$ be, respectively, bounded upper and lower semicontinuous real-valued functions on Q. We seek to characterize $p_0 \in P$ which minimizes the expression

$$\max\{\max_{q \in Q} [f^+(q) - p(q)], \max_{q \in Q} [p(q) - f^-(q)]\}.$$

Remark 11. If $f^+(q) \geqslant f^-(q)$, $q \in Q$, then the expression we are seeking to minimize is equivalent to

$$\max\{\max_{q \in Q} |f^+(q) - p(q)|, \max_{q \in Q} |f^-(q) - p(q)|\}.$$

Note that this equivalence can be used to simplify the expression in [3, Theorem 2].

Remark 12. With $[S(Q)]^*$, B^* and $H^0(Q)$ defined as in Section 5, and $K=H^0(Q)$ we define U on $H^0(Q)$ by

$$U(L_q)=f^+(q); \qquad U(-L_q)=-f^-(q), \qquad q\in Q.$$

Then U is an upper semicontinuous function on $H^0(Q)$.

Proof. As in Remark 10,

$$U^+(L_q)\leqslant \inf_{N^+\in N^+(L_q)} \sup_{L\in N^+} U(L)\leqslant U^+(L_q).$$

But the center term, as in Remark 9, is $f^+(q)$.

Hence

$$U^+(L_q)=U(L_q), \qquad q\in Q$$

and similarly

$$U^+(-L_q)=U(-L_q), \qquad q\in Q.$$

Thus $U^+=U$ on $H^0(Q)$. Hence the result.

Remark 13.

$$\begin{aligned}
&\max\{\max_{q\in Q}[f^+(q)-p(q)], \max_{q\in Q}[p(q)-f^-(q)]\}\\
&\quad=\max\{\max_{q\in Q}[U(L_q)-p(L_q)], \max_{q\in Q}[U(-L_q)-p(-L_q)]\}\\
&\quad=\max_{L\in H^0(Q)}[U(L)-p(L)].
\end{aligned}$$

Theorem 1 now takes the following form:

THEOREM 6. *Let Q be a compact Hausdorff space, f^+ and $-f^-$ be two bounded upper semicontinuous functions on Q and P a linear subspace of $C(Q)$ with $p_0\in P$. Then*

$$\begin{aligned}
E&:=\max\{\max_{q\in Q}[f^+(q)-p_0(q)], \max_{q\in Q}[p_0(q)-f^-(q)]\}\\
&=\inf_{p\in P}\max\{\max_{q\in Q}[f^+(q)-p(q)], \max_{q\in Q}[p(q)-f^-(q)]\}
\end{aligned}$$

if and only if for each $p\in P$ there exists $q\in Q$ such that either

(i) $f^+(q)-p_0(q)=E$ *and* $p(q)\leqslant 0$; *or*

(ii) $p_0(q)-f^-(q)=E$ *and* $p(q)\geqslant 0$.

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